THE LOGISTIC EQUATION: SOLUTIONS AND DEMOGRAPHIC INTERPRETATION

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Abstract

The present paper deals with demographically important solutions to the fundamental logistic equation. Besides the common one, the reversed, the over-growth and the supra-extinction logistic models are briefly analyzed, pointing out their specific properties.

Among the "aberrant" logistic models, the supra-extinction one, which might correspond to an involution of population soon after a nuclear War, seams to be the most dramatic. Indeed, compared to the other normal logistic branches, the demographic recovery needs more than 18 years, just for entering the over-growth phase, necessary in attending the new equilibrium stage.

Keywords: *Logistic equation, Demography, Logistics of Population Dynamics.*

The S-shaped logistic function was applied for the first time, in the first half of the 19th century, by Pierre Francois Verhulst, to the study of population evolution. He put into evidence, after an exponential growth, a slowing down of the process, which is the result of saturation, followed by stopping, at maturity. Such a result represented the basis for the prey-predator model – describing the population dynamics in ecological systems -, independently elaborated by Alfred James Lotka and Vito Volterra.

In time, the logistic function acquired more complicated forms, nowadays being applicable in a large range of domains, such as: biology, medicine, biomathematics, neuronal networks, demography, economy, chemistry, physics, statistics, social and political sciences.

The fundamental equation of the logistic model

Let us have equation:

$$\frac{dN}{dt} = \lambda N + \nu N^2 \qquad (1)$$

where:

t is time;

N - number of individuals;

$\lambda = \frac{ln2}{T_{1/2,2}}$, with

. $\lambda < 0$: $|\lambda| = \frac{ln2}{T_{1/2}}$, $T_{1/2}$ = mean life time of population in free state;

. $\lambda > 0$: $\lambda = \frac{ln2}{T_2}$, T_2 = mean time of population doubling, in free state,

$\nu = \frac{1}{T_0 N_c}$, with T_0: medium period of accessing or producing resources,

N_c: average, characteristic number of individuals from a healthy population.

In principle, the solutions with $N > 0$ (positive) expresses the number of rich people (from the middle class upwards), while those with $N < 0$ give the number of the poor ones (financially, from the middle class downwards).

The determination method of the general solution of equation (1), independent on the particular signs "+" or " " of parameters an , is described in the following.

Consequently, from:

$$\frac{1}{N^2}\frac{dN}{dt} = \frac{\lambda}{N} + \nu \Rightarrow \frac{d}{dt}\left(\frac{1}{N}\right) = -\frac{\lambda}{N} - \nu,$$

using notation:

$$z = \frac{1}{N} \ : \ \frac{dz}{dt} = -\lambda z - \nu,$$

there results, through *variation* of constant K

from the expression of solution:

$$z(t) = K(t)e^{-\lambda t},$$

the string of expressions:

$$e^{-\lambda t}\frac{dK}{dt} = -\nu \rightarrow \frac{dK}{dt} = -\nu e^{\lambda t} \Rightarrow \quad K(t) = K_0 - \frac{\nu}{\lambda}e^{\lambda t},$$

that is, obtaining, for $z = 1/N(t)$, the general solution (independent on sign):

$$z(t) = \frac{1}{N(t)} = K_0 e^{-\lambda t} - \frac{\nu}{\lambda} \Rightarrow \boxed{N(t) = \frac{1}{K_0 e^{-\lambda t} - \frac{\nu}{\lambda}}}$$

Canonically, the first criterion for the establishment of the particular solutions is given by condition Cauchy: $N(t=0) = N_0$, so that $K_0 = \frac{1}{N_0} + \frac{\nu}{\lambda}$, which means:

$$N(t) = \frac{1}{\left(\frac{1}{N_0} + \frac{\nu}{\lambda}\right)e^{-\lambda t} - \frac{\nu}{\lambda}} \Rightarrow \boxed{N(t) = \frac{\lambda/\nu}{\left(1 + \frac{\lambda/\nu}{N_0}\right)e^{-\lambda t} - 1}} \quad (2)$$

The second criterion is represented by the concrete signs, "+" or "–", of the two control parameters λ, and ν; evidently, the following situations – mentioned according to their importance in the literature of the field – are possible:

I. Common logistic model, $\lambda > 0, \ \nu < 0$;

II. Reversed logistic model, $\lambda < 0, \ \nu > 0$;

III. Over-growth logistic model, $\lambda > 0, \ \nu > 0$;

IV. Supra-extinction logistic model, $\lambda < 0, \ \nu < 0$.

Analysis of each of these models will be performed in the following, stress being laid on their really specific properties.

I. THE COMMON LOGISTIC MODEL

It corresponds to an – economically and socially - middle class population with a natural growth rate, $\lambda > 0$, which, nevertheless, globally, consumes more than it produces, so that $\nu = -|\nu| < 0$.

Under such circumstances, the concrete form of the solution of logistic equation (2) becomes:

$$N(t) = \frac{N_* e^{\lambda t}}{e^{\lambda t} - \left(1 - \frac{N_*}{N_0}\right)}, \quad \text{where} \quad N_* = \frac{\lambda}{|\nu|} > 0$$

represents the equilibrium value.

As one may observe, if $0 < N_0 < N_*$, that is the population is "individually" prosperous predominant is the number of those "who have" comparatively with those who "do not have all they need" yet the initial effective number (*i.e.*, at $t = 0$), N_0, does not exceed the equilibrium value:

$$N_* = (ln2)\frac{T_0}{T_m}N_c,$$

then, as $1 - \frac{N_*}{N_c} < 0$, the whole solution becomes:

$$N(t) = N_* \frac{e^{\lambda t}}{e^{\lambda t} + \left(\frac{N_*}{N_0} - 1\right)}, \quad (3)$$

being regular (that is, lacking $\pm\infty$ singularities) and **positive** over the whole real axis $t \in (-\infty, +\infty)$.

Indeed, even if for $t \rightarrow -\infty$, which means for quite a long time, the number of prosperous individuals, $N > 0$, was very low, because of the **low** natural birth rate and consumption, $|\nu| N^2 < \lambda N$, this number enters a quasi-linear growth process which will end when consumption will increase yet without exceeding, in remote future $t \rightarrow \infty$, value N_*. This is the "absolutely" stable logistic branch.

However, if, meaning that the population exceeds, as number of individuals (in a certain given moment, here selected as $t_0 = 0$) the equilibrium value N_*, then the second term from the denominator of (3) reverses the sign, which becomes negative, and the solution gets the form:

$$N(t) = N_* \frac{e^{\lambda t}}{e^{\lambda t} - \left(1 - \frac{N_*}{N_0}\right)}, \text{ with } 0 < \frac{N_*}{N_0} < 1. \quad (4)$$

which evidences the second important characteristic of the common logistic model, namely its **self-regulation** capacity. Even if, in free state, being not influenced by the access to the limited subsistence resources, the population would have a natural growth, λN, when becoming supra-critical ($N_0 > N_*$), the exaggerated consumption rate, $-|\nu| N^2$, exceeds the natural growth one, so that, as a matter of fact, the effective number of individuals, that is N, begins to decrease:

$$\frac{dN}{dt} = \lambda N \left(1 - \frac{|\nu|}{\lambda} N\right) = \lambda N \left(1 - \frac{N}{N_*}\right) < 0, \text{ for}$$

$$N > N_*.$$

Just on the line, for $t \to \infty$, that is in remote future, one may obtain:

$$N(\infty) = lim_{t \to \infty} \frac{N_* e^{\lambda t}}{e^{\lambda t} - \left(1 - \frac{N_*}{N_0}\right)} \cong N_* + N_* \left(1 - \frac{N_*}{N_0}\right) e^{-\lambda t} = N_* + 0_+,$$

meaning re-establishment of the equilibrium value N_* (on the superior branch) 0_+.

More than that, in this case ($N_0 > N_*$), the model shows one more thing, namely that the solution **cannot** be indefinitely prolonged in the past, namely for $t \in (-\infty, 0]$, because a (singular) critical moment exists:

$$t_c = \frac{1}{\lambda} ln \left[1 - \frac{N_*}{N_0}\right] < 0,$$

when the denominator of solution (4) is vanished (which is not the case of the numerator) and, consequently, expressed in scientific terms, the solution "explodes":

$$N(t_c + 0_+) \to +\infty, \text{ and } N(t_c - 0_+) \to -\infty.$$

Such aspect of unbounded discontinuity of the solution is, obviously, a shortcoming of the model, as the control parameters, λ and $\nu = -|\nu|$, were taken as constant. Actually, other dynamic non-linear effects also exist, even if not considered here, such as voluntary birth control – that is the value and sign of parameter λ – and especially the propensity of the rich people of investing for higher productivity, that is for changing parameter ν from a consumption one, $\nu = -|\nu| < 0$, into a production one, $\nu > 0$.

Anyway, supra-critical logistic models with constant parameters announce authentic social and economic splits.

II. THE REVERSE LOGISTIC MODEL

This case refers to a sufficiently mature population, also possessing the corresponding welfare for investing in productivity, that is $\nu > 0$, but, unfortunately, sufficiently aged for providing any natural increase λ, any more, so that, actually $\lambda = -|\lambda| < 0$. Consequently, the logistic equation (1) becomes:

$$\frac{dN}{dt} = -|\lambda| N + \nu N^2, \text{ with } \nu > 0, \quad (5)$$

two conclusions being already possible on the global behavior of solution $N(t) > 0$. Thus, if giving in (5), $-|\lambda| N$ as a factor, one obtains:

$$\frac{dN}{dt} = -|\lambda| N \left[1 - \frac{N}{N_*}\right],$$

which shows that, for all cases with $N(t_0) < N_*$, the sign of the derivate is **negative** so that, further on, for $t > t_0$, the number of individuals registers a ***continuous* decrease,** the concrete form of the solution being given by expression:

$$N(t) = \frac{N_*}{1 + \left(\frac{N_*}{N_0} - 1\right) e^{|\lambda| t}} \text{, with } \frac{N_*}{N_0} > 1.$$

Indeed, for $t \to \infty$, there results the limit:

$$N(\infty) = lim_{t\to\infty} \frac{N_* N_0}{N_* - N_0} e^{-|\lambda| t} = 0_+,$$

which means that a population with a **negative** birth growth and a **lower** number than the equilibrium value, will be **extinguished** because a lower and lower fraction of it, $\nu N^2 \to 0_+$, contributes to its subsistence.

As a first recovery step from such an extinction, this type of population needs an infusion of young, disciplined (and qualified) couples from the outside, for maintaining productivity and for transforming parameter λ into a **positive** one, thus entering a local over-growth phase. Further on, when the equilibrium value, N_*, is exceeded, **consumption**, $\nu > 0 \to \nu < 0$, should be intensified, for avoiding supra-production. The common logistic model, with $N(t) > N_*$, is thus obtained, seen as tending towards the equilibrium value N_*. As already known, this was the case of Switzerland and of the northern countries in the second half of the XX-th century.

When mentioning the term of "supra-production", in the case $N(t_0) > N_*$, it may be observed that:

$$\frac{dN}{dt} = -|\lambda| N \left[1 - \frac{N}{N_*}\right] = |\lambda| N \left[\frac{N}{N_*} - 1\right] > 0 ,$$

which means entering a process of population increase, described by solution:

$$N(t) = \frac{N_*}{1 - \left(1 - \frac{N_*}{N_0}\right) e^{|\lambda| t}} , \quad \text{with} \quad 0 < \frac{N_*}{N_0} < 1,$$

which announces the infinitesimal, critical moment:

$$t_c = \frac{1}{|\lambda|} ln\left[\frac{1}{1 - N_* / N_0}\right] > 0,$$

prior to which, that is for $t = t_c - 0_+$, one has:

$$N(t_c - 0_+) = lim_{\varepsilon\to 0_+} \frac{N_*}{|\lambda| \varepsilon} = +\infty.$$

From a practical perspective, such a result is unacceptable – representing an insufficiency of the reversed logistic model, with constant parameters as, with the increase of population, the consumption rate will necessarily increase. Consequently, parameter ν becomes negative – expressing consumption , which means entering a local supra-extinction phase, until $N = N_* - 0_+$, obviously followed by a natural increase phase, $\lambda > 0$, for attaining the region of absolute stability of the common logistic model.

As retrospectively observed, it was this the exact strategy of the USA after World War II, namely supra-production until the 55'ies, stimulation of consumption until about 1969 and massive acceptance of immigrants between 1970 and 1980, after which the governmental program *Visa Lottery* was launched.

III. THE OVER-GROWTH MODEL

Together with the supra-extinction one, they form the "aberrant" logistic models, that is, as long as parameters λ and ν are constant, having the same sign, with reference to the canonic solutions with $N > 0$, the possibility of self-adjustment (produced in the case of the authentic logistic model) exists **no more**. That is why, starting with N_0 positive, the corresponding solutions **cannot be** indefinitely prolonged, any longer, over the whole real time axis, socio-economic splits, similar to the (financial) crashes and/or revolutions, thus occurring as an interpretation.

Utilization of such models assumes precisely that, if parameters λ, ν and N_0 may be quite precisely determined by (global) statistical processing, then the critical moments, t_c, may be anticipated, so that corresponding measures for avoiding them shall be taken.

For example, in the case of the over-grow model, with $(\lambda,\nu)>0$, the corresponding solution takes expression:

$$N(t)=\frac{N_* e^{\lambda t}}{\left(1+\frac{N_*}{N_0}\right)-e^{\lambda t}}, \quad \text{with } N_0>0,$$

which shows that, by vanishing the denominator, the moment of crisis, t_c, is given by expression:

$$t_c=\frac{1}{\lambda}ln\left[1+\frac{N_*}{N_0}\right]>0.$$

If, in the initial moment, $t_0=0$, the population is much sub-critical, that is $N_0<<N_*$, then:

$$1+\frac{N_*}{N_0}\cong\frac{N_*}{N_0} \text{ so that:}$$

$$t_c^{(sub)}\cong\frac{ln(N_*/N_0)}{ln2}T_2>T_2.$$

Consequently, it is sufficient – time at least at the level of a generation – for modifying preferentially parameter ν, into a consumption one, $\nu<0$, so that the system will enter the usual, self-adjusting, logistic phase, and the total number of individuals will asymptotically tend towards the equilibrium value N_*.

If, however, at the same moment, $t_0=0$, the society was considerably supra-critical, that is $N_0>>N_*$, then $ln\left[1+\frac{N_*}{N_0}\right]\cong\frac{N_*}{N_0}$ and, consequently, because $N_*=\lambda T_0 N_c$,

$$t_c^{(supra)}\cong\frac{N_c}{N_0}T_0<T_0.$$

Therefore, even if taking $T_0\cong10$ years and $\frac{N_c}{N_0}\cong\frac{1}{10}$, the crisis will be very soon installed, namely:

$$t_c^{(supra)}\cong1\,\text{year}.$$

As the mentioned data are similar to those of Japan, one may speculate (at least, that is without any claim of anticipation) that, if this country will witness overgrowth, it would face, in no more than one-two years, a supra-production crisis. This explains why – for avoiding a crisis – Japan changes its technological lines at intervals of about 10 years, the most frequently used word in their commercial advertisements being *atarashii*, meaning "new", which aims at a periodical increase of the internal consumption.

IV. THE SUPRA-EXTINCTION MODEL

Considering parameters $\lambda=-|\lambda|$ and $\nu=-|\nu|$ constant and negative corresponds to the **involution** of a population (immediately) after a nuclear global conflict, when the number of survivors, N_0, is low (comparatively with the normal, characteristic standard, N_c), while the equilibrium value, $N_*=(ln2)\frac{T_0}{T_{1/2}}N_c$, is high, because of the half time period $T_{1/2}$ of the population, which becomes much lower, as a result of radiations, than the average, normal life time of the (initially) healthy individuals.

Consequently, strictly mathematically, that is without any re-establishment measures, the solution of the supra-extinction model takes the form:

$$N(t)=\frac{N_*}{\left(1+\frac{N_*}{N_0}\right)e^{|\lambda|t}-1},$$

evidencing that, for $\frac{N_*}{N_0}>>1$ and $|\lambda|t\geq1$, it becomes:

$$N(t) \cong N_0 e^{-|\lambda| t},$$

corresponding, as $|\lambda|$ is large, to a rapid exponential extinction, the surviving one **included**.

To a**void** such an aspect, as long as the survivors are still able to work, a first measure to be taken is to move them "in underground" locations. In this way, as a result of the terrestrial screening to radiations, coefficient $\lambda = -|\lambda|$ is brought to zero – which means statistical suppression of mortality – and, necessarily reversion of the sign of the consumption rate into a positive one – of micro-climateric agrarian production – so that the remaining collectivity to enter, statistically speaking, upon the only ascending branch available to it in such moments, namely the solution of the degenerated logistic equation:

$$\frac{dP}{dt} = \nu\ P^2 \ \Rightarrow\ P(t) = \frac{P_0}{1 - \nu P_0 t}.$$

Here, P represents the population capable of supporting the community, as both maintaining of productivity and increase of birth rate. Once known that, after a nuclear cataclysm, both P_0 and ν are low, their product with t is much sub-unitary, $\nu P_0 t << 1$, which leads to a quasi-linear **evolution** phase.

$$P(t) \cong P_0\left[1 + (\nu P_0)t\right] \ \Rightarrow\ \frac{dN}{dt} \cong \nu\ P_0^2 = const. > 0.$$

At (normal) biological scale, and considering the subsequent (socio-economic) self-supporting ability of the new-born ones, an average period of $\Delta t \cong 18$ years is here involved, resulting from $\frac{1}{2}(16 + 20)$, after which the natural increase λ becomes positive, which is the beginning of an over-growth phase, necessary for the rapid attainment of a new equilibrium level, $N_*^{(new)}$, after which, by stimulating consumption, to enter the usual logistic branch, which is absolutely stable to any subsequent fluctuations.

The already observable conclusion is that recovery after a large-scale nuclear catastrophe is the most difficult one among all normal logistic branches, which explains why the nuclear arming during the Cold War was more a menace and not a reality announcing a real aggression.

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